# When is it the case that one tableau suffices? A note on Prince (2015)*

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**Abstract** Prince (2015) shows that OT has the following surprising property: no ranking information is lost by replacing two violation tableaux with the single tableau obtained by summing any row of one of the two original tableaux with any row of the other. Which properties of OT are responsible for this result? This squib answers this question: Prince's result that "one tableau suffices" holds whenever grammatical optimization is relative to a strict additive weak ordering. As an application, Prince's result is extended from OT to HG.

## 1 Introduction

Consider two OT violation tableaux $A$ and $B$ corresponding to the same constraint set $C_1, \ldots, C_n$. They correspond to different underlying forms and list all possible candidates together with the corresponding number of constraint violations. Let **a** be the generic candidate of the former tableau and **b** the generic candidate of the latter, as in (1).

(1) A =

| | $C_1$ | $\cdots$ | $C_k$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|---|
| $\vdots$ | | $\ddots$ | | $\cdot^{\cdot^{\cdot}}$ | |
| ☞ **a** | | | $a_k$ | | |
| $\vdots$ | | $\cdot^{\cdot^{\cdot}}$ | | $\ddots$ | |

B =

| | $C_1$ | $\cdots$ | $C_k$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|---|
| $\vdots$ | | $\ddots$ | | $\cdot^{\cdot^{\cdot}}$ | |
| ☞ **b** | | | $b_k$ | | |
| $\vdots$ | | $\cdot^{\cdot^{\cdot}}$ | | $\ddots$ | |

Let $A+B$ be the tableau defined as follows: for each row **a** of $A$ and each row **b** of $B$, the tableau $A+B$ has a row $\mathbf{a}+\mathbf{b}$ whose entries are the sum $a_k+b_k$ of the corresponding entries $a_k$ and $b_k$ in the two rows **a** and **b**, as in (2). In other words, if constraint $C_k$ assigns two violations to **a** and three violations to **b**, then it assigns five violations to $\mathbf{a}+\mathbf{b}$. The number of rows of the resulting tableau $A+B$ is thus the product of the numbers of rows of the two original tableaux $A$ and $B$.

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(2) A + B =

| | $C_1$ | $\cdots$ | $C_k$ | $\cdots$ | $C_n$ |
|---|---|---|---|---|---|
| $\vdots$ | | $\ddots$ | | $\iddots$ | |
| ☞ $\mathbf{a}+\mathbf{b}$ | | | $a_k+b_k$ | | |
| $\vdots$ | | $\iddots$ | | $\ddots$ | |

Prince (2015) establishes the following theorem 1. Intuitively, it says that any two OT tableaux $A$ and $B$ as in (1) convey exactly the same ranking information as their sum tableau $A+B$ in (2). In other words, "one tableau suffices". This result is surprising, because summation would intuitively be expected to lose information. See Prince (2015, section 2) for discussion of the linguistic implications of this result.

**Theorem 1 ("One OT tableau suffices"; Prince 2015)** *A ranking over the constraint set* $\{C_1,\ldots,C_n\}$ *declares* $\mathbf{a}$ *an OT winner of tableau A and* $\mathbf{b}$ *an OT winner of tableau B if and only if it declares* $\mathbf{a}+\mathbf{b}$ *an OT winner of tableau* $A+B$. ■

This squib addresses the following question: Which of the many specific properties of the OT framework are responsible for Prince's result? In other words, what is the minimal structure required by an optimization-based grammatical formalism in order to satisfy Prince's condition that "one tableau suffices"? Section 2 formulates this question explicitly. Section 3 introduces additive strict weak orderings. And section 4 shows that they provide the minimal structure which suffices to derive Prince's result. As an application, section 5 re-derives Prince's theorem 1 for OT and extends it to the following analogous theorem 2 for HG.

**Theorem 2 ("One HG tableau suffices")** *A weight vector for the constraint set* $\{C_1,\ldots,C_n\}$ *declares* $\mathbf{a}$ *a HG winner of tableau A and* $\mathbf{b}$ *a HG winner of tableau B if and only if it declares* $\mathbf{a}+\mathbf{b}$ *a HG winner of tableau* $A+B$. ■

## 2 Formulating the question

Consider the collection of all tuples $\mathbf{a}=(a_1,\ldots,a_n)$ of $n$ numbers, usually denoted by $\mathbb{R}^n$. Let $\mathbf{a}+\mathbf{b}=(a_1+b_1,\ldots,a_n+b_n)$ be the component-wise sum between any two of these tuples $\mathbf{a}=(a_1,\ldots,a_n)$ and $\mathbf{b}=(b_1,\ldots,b_n)$. Let $A\oplus B=\{\mathbf{a}+\mathbf{b}\,|\,\mathbf{a}\in A,\mathbf{b}\in B\}$ be the *Minkowski sum* of any two sets $A,B\subseteq\mathbb{R}^n$ of tuples, which consists of the sum of an arbitrary tuple from $A$ and an arbitrary tuple from $B$. Consider a strict partial order $<$ among the tuples of $\mathbb{R}^n$.[1] Given a set $A\subseteq\mathbb{R}^n$, let $\min_<(A)$ be

1 This means that $<$ satisfies the following three axioms for any tuples $\mathbf{a},\mathbf{b},\mathbf{c}$ in $\mathbb{R}^n$: it is *irreflexive*, namely $\mathbf{a}\not<\mathbf{a}$; it is *asymmetric*, namely $\mathbf{a}<\mathbf{b}$ entails $\mathbf{b}\not<\mathbf{a}$; and it is *transitive*, namely $\mathbf{a}<\mathbf{b}$ and $\mathbf{b}<\mathbf{c}$ entail $\mathbf{a}<\mathbf{c}$.

the collection of its *minimal elements*. This means that $\widehat{\mathbf{a}}$ belongs to $\min_<(A)$ if and only if $\widehat{\mathbf{a}}$ belongs to $A$ and furthermore no element $\mathbf{a}$ of $A$ is smaller than $\widehat{\mathbf{a}}$ according to $<$, as stated in (3).

(3) $\min\limits_<(A) = \left\{\widehat{\mathbf{a}} \in A \,\middle|\, \text{there exists no } \mathbf{a} \in A \text{ s.t. } \mathbf{a} < \widehat{\mathbf{a}}\right\}$

A tuple $\mathbf{a} = (a_1, \ldots, a_n)$ can be interpreted as the violation profile assigned to a candidate by $n$ constraints: $a_1$ is the number of violations assigned by constraint $C_1$, $a_2$ is the number of violations assigned by constraint $C_2$, and so on. Thus, a set $A \subseteq \mathbb{R}^n$ of tuples can be interpreted as a violation tableau. We consider an *optimization-based* grammatical framework whereby a grammar corresponds to a certain (possibly partial) strict order $<$ on $\mathbb{R}^n$ and thus declares *optimal* those candidates of a tableau $A$ which have a $<$-minimal violation profile, as stated in (4); see section 5 for concrete examples.

(4) optimal candidates in the tableau $A = \min\limits_<(A)$

Prince's condition that a single violation tableau suffices thus means that the following identity (5) holds for any two sets $A, B \subseteq \mathbb{R}^n$. This identity says in turn that the two operations of sum and minimum commute: by first summing the two sets together and then taking the minimum (as prescribed by the left-hand side) we get the same result that we get by first taking the minima of the two sets and then summing them together (as prescribed by the right-hand side).

(5) $\boxed{\min\limits_<(A \oplus B) = \min\limits_<(A) \oplus \min\limits_<(B)}$

This squib addresses the following question: what are the minimal assumptions on the strict partial order $<$ which ensure that Prince's identity (5) holds?

## 3 Additive weak orderings

The implication (6) captures the intuitive idea that, if $\mathbf{a}$ is smaller than $\mathbf{b}$ and if the same quantity $\mathbf{c}$ is added to both, the resulting sum $\mathbf{a}+\mathbf{c}$ ought to be smaller than the sum $\mathbf{b}+\mathbf{c}$. Although intuitive, this implication does not follow from the definition of strict order: it is possible to construct strict orders which fail at (6). Thus, a (possibly partial) strict order $<$ on $\mathbb{R}^n$ is called *additive* relative to the component-wise sum operation $+$ provided the implication (6) holds for any tuples $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of $\mathbb{R}^n$(Anderson & Feil 1988).

(6) If: $\mathbf{a} < \mathbf{b}$
then: $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$

Suppose that the strict order $<$ is partial, not total. This means that there exists some pair of elements $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ such that $\mathbf{a} \not< \mathbf{b}$ and $\mathbf{b} \not< \mathbf{a}$. In this case, we say that $\mathbf{a}$ and

$\mathbf{b}$ are $<$-*incommensurable* and we write $\mathbf{a} \sim \mathbf{b}$. In other words, every partial strict order defines a corresponding *incommensurability relation* $\sim$, as in (7).

(7) $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a} \not< \mathbf{b}$ and $\mathbf{b} \not< \mathbf{a}$

If the order $<$ is additive and thus satisfies the implication (6), the corresponding incommensurability relation $\sim$ is additive as well, namely it satisfies the analogous implication (8) for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. In fact, suppose by contradiction that (8) fails, namely that $\mathbf{a} \sim \mathbf{b}$ but $\mathbf{a}+\mathbf{c} \not\sim \mathbf{b}+\mathbf{c}$. The latter condition $\mathbf{a}+\mathbf{c} \not\sim \mathbf{b}+\mathbf{c}$ means that either $\mathbf{b}+\mathbf{c} < \mathbf{a}+\mathbf{c}$ or $\mathbf{a}+\mathbf{c} < \mathbf{b}+\mathbf{c}$; for concreteness, suppose the former case holds. By adding $-\mathbf{c}$ at both sides, $\mathbf{b}+\mathbf{c} < \mathbf{a}+\mathbf{c}$ entails $\mathbf{b} < \mathbf{a}$ by virtue of (6). The conclusion $\mathbf{b} < \mathbf{a}$ thus obtained contradicts the assumption that $\mathbf{a} \sim \mathbf{b}$.

(8) If: $\mathbf{a} \sim \mathbf{b}$
then: $\mathbf{a}+\mathbf{c} \sim \mathbf{b}+\mathbf{c}$

Since $<$ is by assumption a strict order, then $\mathbf{a} \not< \mathbf{a}$ for any tuple $\mathbf{a}$ of $\mathbb{R}^n$. In other words, any element $\mathbf{a}$ is incommensurable with itself and the incommensurability relation $\sim$ is thus reflexive. Furthermore, $\sim$ is obviously symmetric—as symmetry is built explicitly into its definition (7). The strict order $<$ is called a *weak ordering* provided the corresponding incommensurability relation $\sim$ is also transitive, namely qualifies as an equivalence relation over $\mathbb{R}^n$ (Roberts & Tesman 2005, section 4.2.4).

We now have two assumptions on the strict partial order $<$: that it is *additive* and that it is a *weak ordering*. Section 4 will show that these two assumptions are necessary and sufficient to guarantee Prince's identity (5). Towards establishing that result, the rest of this section studies closely the combination of these two assumptions. To start, the following lemma provides an equivalent characterization of additive weak orderings on $\mathbb{R}^n$ in terms of the implication (9). This implication says that the validity of an inequality is not affected by adding incommensurable elements at both sides. The intuition behind this implication (9) can be brought out as follows. The assumption that $<$ is additive means that it satisfies the implication (6). This implication can be restated as follows: if $\mathbf{a} < \mathbf{b}$ and $\mathbf{c} = \mathbf{d}$, then $\mathbf{a}+\mathbf{c} < \mathbf{b}+\mathbf{d}$. Since $<$ is a weak ordering, its corresponding incommensurability relation $\sim$ is an equivalence relation, namely $\sim$ generalizes the identity $=$. The implication (9) is thus a natural generalization of the implication (6) used to define additivity.

**Lemma 1** *Consider a strict (possibly partial) order $<$ over $\mathbb{R}^n$. Then, $<$ is an additive weak ordering if and only if the following implication*

*(9) If:* $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{c} \sim \mathbf{d}$
*then:* $\mathbf{a}+\mathbf{c} < \mathbf{b}+\mathbf{d}$

*holds for any tuples* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* $\mathbb{R}^n$. ■

*Proof.* Let me show that (9) entails that $<$ is an additive weak ordering. As noted above, the incommensurability relation $\sim$ is reflexive, namely $\mathbf{c} \sim \mathbf{c}$ for any element $\mathbf{c} \in \mathbb{R}^n$. Thus, $<$ is additive because the implication (6) is a special case of the implication (9). Next, let me show that the incommensurability relation $\sim$ is transitive. Assume by contradiction that were not the case, namely that $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{b} \sim \mathbf{c}$ but $\mathbf{a} \not\sim \mathbf{c}$. The latter condition $\mathbf{a} \not\sim \mathbf{c}$ means that either $\mathbf{a} < \mathbf{c}$ or $\mathbf{c} < \mathbf{a}$; for concreteness, suppose the former case holds. By (9), $\mathbf{a} < \mathbf{c}$ and $\mathbf{c} \sim \mathbf{b}$ entails $\mathbf{a} + \mathbf{c} < \mathbf{c} + \mathbf{b}$. By (9) again, $\mathbf{a} + \mathbf{c} < \mathbf{c} + \mathbf{b}$ and $-\mathbf{c} \sim -\mathbf{c}$ (which holds because $\sim$ is reflexive), entails $\mathbf{a} < \mathbf{b}$. The conclusion $\mathbf{a} < \mathbf{b}$ contradicts the hypothesis $\mathbf{a} \sim \mathbf{b}$.

Let me now show that the assumption that $<$ is an additive weak ordering entails (9). Assume by contradiction that (9) fails, namely that $\mathbf{a} < \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$ but $\mathbf{a} + \mathbf{c} \not< \mathbf{b} + \mathbf{d}$. The latter means that either $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{d}$ or $\mathbf{b} + \mathbf{d} < \mathbf{a} + \mathbf{c}$. Let me consider the two cases separately, starting with the former case. The hypothesis (6) that $<$ is additive entails the conclusion (8) that $\sim$ is additive as well. By (8), $\mathbf{c} \sim \mathbf{d}$ entails $\mathbf{c} + \mathbf{b} \sim \mathbf{d} + \mathbf{b}$. By the hypothesis that $\sim$ is transitive, this conclusion $\mathbf{c} + \mathbf{b} \sim \mathbf{d} + \mathbf{b}$ plus the contradictory assumption $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{d}$ entail that $\mathbf{a} + \mathbf{c} \sim \mathbf{c} + \mathbf{b}$. Again by (8), $\mathbf{a} + \mathbf{c} \sim \mathbf{c} + \mathbf{b}$ entails $\mathbf{a} \sim \mathbf{b}$ by adding $-\mathbf{c}$ at both sides. The conclusion $\mathbf{a} \sim \mathbf{b}$ contradicts the hypothesis $\mathbf{a} < \mathbf{b}$. Consider next the other contradictory assumption, namely that $\mathbf{b} + \mathbf{d} < \mathbf{a} + \mathbf{c}$. By (6), the hypothesis $\mathbf{a} < \mathbf{b}$ entails $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$. By transitivity of $<$, the latter together with $\mathbf{b} + \mathbf{d} < \mathbf{a} + \mathbf{c}$ entail $\mathbf{b} + \mathbf{d} < \mathbf{b} + \mathbf{c}$. By (6) again, the latter entails $\mathbf{d} < \mathbf{c}$, which contradicts the hypothesis that $\mathbf{c} \sim \mathbf{d}$. □

The implication (9) used in the preceding lemma features the two relations $<$ and $\sim$ simultaneously. The following lemma provides another characterization of additive weak orderings in terms of the two implications (6) and (10), where the two relations $<$ and $\sim$ feature separately. This characterization will be used in establishing the main result of the next section.

**Lemma 2** *Consider a strict (possibly partial) order $<$ over $\mathbb{R}^n$. Then, $<$ is an additive weak ordering if and only if the implication (6) together with the following implication*

*(10)* *If:* $\mathbf{a} \sim \mathbf{b}$ *and* $\mathbf{c} \sim \mathbf{d}$;
*then:* $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{d}$

*hold for any tuples* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* $\mathbb{R}^n$. ■

*Proof.* Let me show that (9) entails (10). Assume by contradiction that (10) fails, namely that $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$ but that $\mathbf{a} + \mathbf{c} \not\sim \mathbf{b} + \mathbf{d}$. The latter condition means that either $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{d}$ or $\mathbf{b} + \mathbf{d} < \mathbf{a} + \mathbf{c}$; for concreteness, suppose the former case holds. By (9) and the hypothesis $\mathbf{c} \sim \mathbf{d}$, $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{d}$ entails $(\mathbf{a} + \mathbf{c}) + \mathbf{d} < (\mathbf{b} + \mathbf{d}) + \mathbf{c}$. By (9) again and $-\mathbf{c} - \mathbf{d} \sim -\mathbf{c} - \mathbf{d}$ (because $\sim$ is reflexive), $(\mathbf{a} + \mathbf{c}) + \mathbf{d} < (\mathbf{b} + \mathbf{d}) + \mathbf{c}$ entails

$(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{c}-\mathbf{d} < (\mathbf{b}+\mathbf{d}+\mathbf{c})-\mathbf{c}-\mathbf{d}$. The latter says that $\mathbf{a} < \mathbf{b}$, contradicting the hypothesis that $\mathbf{a} \sim \mathbf{b}$.

Let me now show that (6) and (10) entail (9). Assume by contradiction that (9) fails, namely that $\mathbf{a} < \mathbf{b}$ and $\mathbf{c} \sim \mathbf{d}$ but that $\mathbf{a}+\mathbf{c} \not< \mathbf{b}+\mathbf{d}$. The latter means in turn that either $\mathbf{a}+\mathbf{c} \sim \mathbf{b}+\mathbf{d}$ or $\mathbf{b}+\mathbf{d} < \mathbf{a}+\mathbf{c}$. Let me consider these two cases separately, starting with the former case. By (10) and the hypothesis $\mathbf{c} \sim \mathbf{d}$, $\mathbf{a}+\mathbf{c} \sim \mathbf{b}+\mathbf{d}$ entails $(\mathbf{a}+\mathbf{c})+\mathbf{d} \sim (\mathbf{b}+\mathbf{d})+\mathbf{c}$. By (10) again and $-\mathbf{c}-\mathbf{d} \sim -\mathbf{c}-\mathbf{d}$ (because $\sim$ is reflexive), $(\mathbf{a}+\mathbf{c})+\mathbf{d} \sim (\mathbf{b}+\mathbf{d})+\mathbf{c}$ entails $(\mathbf{a}+\mathbf{c}+\mathbf{d})-\mathbf{c}-\mathbf{d} \sim (\mathbf{b}+\mathbf{d}+\mathbf{c})-\mathbf{c}-\mathbf{d}$. The latter says that $\mathbf{a} \sim \mathbf{b}$, contradicting the hypothesis that $\mathbf{a} < \mathbf{b}$.

Consider next the other contradictory assumption, namely that $\mathbf{b}+\mathbf{d} < \mathbf{a}+\mathbf{c}$. By (6), the hypothesis $\mathbf{a} < \mathbf{b}$ entails $\mathbf{a}+\mathbf{d} < \mathbf{b}+\mathbf{d}$. By transitivity of $<$, $\mathbf{b}+\mathbf{d} < \mathbf{a}+\mathbf{c}$ and $\mathbf{a}+\mathbf{d} < \mathbf{b}+\mathbf{d}$ entail $\mathbf{a}+\mathbf{d} < \mathbf{a}+\mathbf{c}$. By (6) again, $\mathbf{a}+\mathbf{d} < \mathbf{a}+\mathbf{c}$ entails $\mathbf{d} < \mathbf{c}$, contradicting the hypothesis that $\mathbf{c} \sim \mathbf{d}$. □

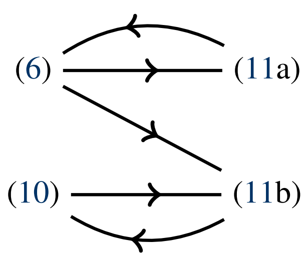
## 4 Answering the question

This section proves lemma 3, which is the main result of this squib. The "if" statement of the lemma says that the assumption that $<$ is an additive weak ordering provides *sufficient* structure for Prince's identity (5) to hold, ensuring that "one tableau suffices". Furthermore, the "only if" statement of the lemma says that the assumption that $<$ is an additive weak ordering provides *necessary* structure for Prince's identity (5) to hold. In other words, this lemma solves the question raised in section 2. Towards proving the lemma, it useful to split Prince's identity (5) into the two inclusions (11).

(11) a. $\min(A \oplus B) \subseteq \min(A) \oplus \min(B)$
b. $\min(A \oplus B) \supseteq \min(A) \oplus \min(B)$

The proof of the lemma relies on the characterization of additive weak orderings provided by lemma 2 through the two implications (6) and (10). The train of reasoning is summarized in (12). The implication (6) which captures additivity is shown to be equivalent to the first half (11a) of Prince's identity. Given additivity, the implication (10) which captures weak ordering is shown to be equivalent to the second half (11b) of Prince's identity.

(12)



**Lemma 3** *Consider a strict (possibly partial) order $<$ over $\mathbb{R}^n$. Prince's identity (5) holds for any two sets $A, B \subseteq \mathbb{R}^n$ if and only if $<$ is an additive weak ordering.* ■

*Proof.* Let me show that the implication (6) entails the inclusion (11a). Consider an element $\mathbf{x} \in \min(A \oplus B)$. Thus in particular $\mathbf{x} \in A \oplus B$. This means that $\mathbf{x} = \widehat{\mathbf{a}} + \widehat{\mathbf{b}}$, for some $\widehat{\mathbf{a}} \in A$ and some $\widehat{\mathbf{b}} \in B$. Suppose by contradiction that $\widehat{\mathbf{a}} + \widehat{\mathbf{b}} \notin \min(A) \oplus \min(B)$. This means that either $\widehat{\mathbf{a}} \notin \min(A)$ or else $\widehat{\mathbf{b}} \notin \min(B)$. For concreteness, suppose that $\widehat{\mathbf{a}} \notin \min(A)$. This means in turn that there exists $\mathbf{a} \in A$ such that $\mathbf{a} < \widehat{\mathbf{a}}$. By (6), $\mathbf{a} < \widehat{\mathbf{a}}$ entails $\mathbf{a} + \widehat{\mathbf{b}} < \widehat{\mathbf{a}} + \widehat{\mathbf{b}}$. The inequality thus obtained contradicts the hypothesis that $\widehat{\mathbf{a}} + \widehat{\mathbf{b}} \in \min(A \oplus B)$, because $\mathbf{a} + \widehat{\mathbf{b}} \in A \oplus B$ (since $\mathbf{a} \in A$ and $\widehat{\mathbf{b}} \in B$).

Let me show that the inclusion (11a) vice versa entails the implication (6). Suppose that $\mathbf{a} < \mathbf{b}$. Let $A = \{\mathbf{a}, \mathbf{b}\}$ and $B = \{\mathbf{c}\}$. Since $\mathbf{a} < \mathbf{b}$ by hypothesis, then $\min(A) = \mathbf{a}$. Furthermore, since $\min(B) = \mathbf{c}$ (because $B$ is a singleton), then $\min(A) \oplus \min(B) = \{\mathbf{a} + \mathbf{c}\}$. Finally, $A \oplus B = \{\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}\}$. The inclusion (11a) thus becomes $\min\{\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}\} \subseteq \{\mathbf{a} + \mathbf{c}\}$. This means in turn that $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$.

Let me show that the two implications (6) and (10) together entail the other inclusion (11b). Let $\widehat{\mathbf{a}} \in \min(A)$ and $\widehat{\mathbf{b}} \in \min(B)$. Suppose by contradiction that $\widehat{\mathbf{a}} + \widehat{\mathbf{b}} \notin \min(A \oplus B)$. This means that there exists $\mathbf{a} + \mathbf{b} \in \min(A \oplus B)$ such that $\mathbf{a} + \mathbf{b} < \widehat{\mathbf{a}} + \widehat{\mathbf{b}}$. Since $\mathbf{a} \in A$ and $\widehat{\mathbf{a}} \in \min(A)$, either $\widehat{\mathbf{a}} < \mathbf{a}$ or else $\mathbf{a} \sim \widehat{\mathbf{a}}$. Analogously, since $\mathbf{b} \in B$ and $\widehat{\mathbf{b}} \in \min(B)$, either $\widehat{\mathbf{b}} < \mathbf{b}$ or else $\mathbf{b} \sim \widehat{\mathbf{b}}$. If $\widehat{\mathbf{a}} < \mathbf{a}$, then $\widehat{\mathbf{a}} + \mathbf{b} < \mathbf{a} + \mathbf{b}$ by (6), contradicting the assumption that $\mathbf{a} + \mathbf{b} \in \min(A \oplus B)$. If $\widehat{\mathbf{b}} < \mathbf{b}$, I can reason analogously. If $\mathbf{a} \sim \widehat{\mathbf{a}}$ and $\mathbf{b} \sim \widehat{\mathbf{b}}$, then $\mathbf{a} + \mathbf{b} \sim \widehat{\mathbf{a}} + \widehat{\mathbf{b}}$ by (10), contradicting the hypothesis that $\mathbf{a} + \mathbf{b} < \widehat{\mathbf{a}} + \widehat{\mathbf{b}}$.

Finally, let me show that the inclusion (11b) entails the implication (10). Suppose that $\mathbf{a} \sim \mathbf{b}$ and that $\mathbf{c} \sim \mathbf{d}$. Consider the sets $A = \{\mathbf{a}, \mathbf{b}\}$ and $B = \{\mathbf{c}, \mathbf{d}\}$. Incommensurability says that $\min(A) = \{\mathbf{a}, \mathbf{b}\}$ and $\min(B) = \{\mathbf{c}, \mathbf{d}\}$. Hence, $\min(A) \oplus \min(B) = \{\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{d}, \mathbf{b} + \mathbf{d}\} = A \oplus B$. The inclusion (11b) thus says that $A \oplus B = \min(A \oplus B)$. In other words, the elements of $A \oplus B$ are all incommensurable, and thus in particular $\mathbf{a} + \mathbf{c} \sim \mathbf{b} + \mathbf{d}$. □

## 5 Applications

This section presents two applications of lemma 3: first, it re-derives Prince's result that "one tableau suffices" in the case of OT (theorem 1); second, it extends that result to HG (2).

### 5.1 Total orders and OT

Any *total* strict order is trivially a weak ordering, because two elements are incommensurable only if they are identical, so that $\sim$ coincides with the identity and it is therefore transitive. Lemma 3 thus ensures that "one tableau suffices" whenever grammatical optimization is relative to an additive strict total order. As a special case, let $<$ be the *lexicographic order* on $\mathbb{R}^n$, defined as follows: for any two tuples $\mathbf{a} = (a_1, \dots, a_n), \mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$, let $\mathbf{a} < \mathbf{b}$ if and only if there exists $k \in \{1, \dots, n\}$ such that conditions (13) hold.

(13)
$$\begin{aligned} a_1 &= b_1 \\ &\vdots \\ a_k &= b_k \\ a_{k+1} &< b_{k+1} \end{aligned}$$

Obviously, $<$ is a strict total order on $\mathbb{R}^n$. Furthermore, $<$ is additive, namely it satisfies the implication (6): the assumption $\mathbf{a} < \mathbf{b}$ that (13) holds obviously entails the conclusion $\mathbf{a} + \mathbf{c} < \mathbf{b} + \mathbf{c}$ that (14) holds.

(14)
$$\begin{aligned} a_1 + c_1 &= b_1 + c_1 \\ &\vdots \\ a_k + c_k &= b_k + c_k \\ a_{k+1} + c_{k+1} &< b_{k+1} + c_{k+1} \end{aligned}$$

Prince's OT theorem 1 thus follows from lemma 3: one OT tableau suffices because grammatical optimization in OT is computed relative to the lexicographic order, which is additive and total.

## 5.2 Orders defined by additive utility functions and HG

Suppose that $<$ is a partial order over $\mathbb{R}^n$ defined in terms of a *utility function* $U$ from $\mathbb{R}^n$ to $\mathbb{R}$, in the sense that the ordering $\mathbf{a} < \mathbf{b}$ holds between any two tuples $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ if and only if the utility of $\mathbf{a}$ is strictly smaller than the utility of $\mathbf{b}$, namely $U(\mathbf{a}) < U(\mathbf{b})$. Assume furthermore that $U$ is additive, namely that $U(\mathbf{a} + \mathbf{b}) = U(\mathbf{a}) + U(\mathbf{b})$. Equivalently, assume that there exists a *weight vector* $\mathbf{w} = (w_1, \dots, w_n)$ such that $U(\mathbf{a})$ is proportional to the length of the projection of $\mathbf{a}$ on the weight vector $\mathbf{w}$, namely $U(\mathbf{a}) = \sum_{i=1}^n a_i w_i$ for any vector $\mathbf{a} = (a_1, \dots, a_n)$. This order $<$ is partial, as two different vectors can have the same projection on the weight vector $\mathbf{w}$. Yet, the two conditions (6) and (10) are trivially satisfied, ensuring that $<$ is a strict partial additive weak ordering on $\mathbb{R}^n$. Theorem 2 for HG thus follows from lemma 3: "one tableau suffices" in the case of HG because grammatical optimization in HG is computed relative to the ordering induced by an additive utility function, which is an additive weak ordering.

## 6 Conclusion

It is often useful to try to understand the minimal structure which supports a certain formal property. For instance, in the introduction to his recent book, Talagrand (2014) writes: "The practitioner of stochastic processes is likely to be struggling at any given time with his favorite model of the moment, a model that will typically involve a rather rich and complicated structure. There is a near infinite supply of such models. Fashions come and go, and the importance with which we view any specific model is likely to strongly vary over time. The first advice the author received from his advisor Gustave Choquet was as follows: Always consider a problem under the minimum structure in which it makes sense. This advice will probably be fruitful in the future as it has been in the past [...]. By following it, one is naturally led to the study of problems with a kind of minimal and intrinsic structure. Besides the fact that it is much easier to find the crux of the matter in a simple structure than in a complicated one, there are not so many really basic structures, so one can hope that they will remain of interest for a very long time."

Applying Choquet's advice to Prince's result, this squib has addressed the following question: Which of the many specific properties of the OT framework are responsible for Prince's result that "one tableau suffices"? In other words, what is the minimal structure required by an optimization-based grammatical formalism in order to satisfy Prince's condition that "one tableau suffices"? The answer is that additive weak orderings are the minimal structure needed to derive Prince's result.

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